

EXERCISES IN DESCRIPTIVE SET THEORY, II

1. EXERCISE

Suppose X is a Polish space and let $\mathcal{K}(X)$ be the hyperspace of compact subsets of X with the Vietoris topology, i.e., the topology induced by the Hausdorff metric. Suppose $(K_i)_{i \in I}$ is a directed family of compact subsets of X , i.e., for all i, j there is some l such that $K_i \cap K_j \supseteq K_l$. Quasiorder I by $i \leq j$ if $K_i \supseteq K_j$ and show that the net $(K_i)_{i \in I}$ converges to $K = \bigcap_{i \in I} K_i$ in $\mathcal{K}(X)$.

2. EXERCISE

Prove that Σ_2^0 games are determined as follows.

Let $B \subseteq A^{\mathbb{N}}$ be Σ_2^0 and write $B = \bigcup_n F_n$, where $F_n \subseteq A^{\mathbb{N}}$ is closed. Write $F_n = [T_n]$ for a pruned tree T_n on A . We define sets $W_\xi \subseteq A^{<\mathbb{N}}$ by recursion on ξ :

$$s \in W_0 \Leftrightarrow |s| \text{ is even} \ \& \ \exists n \text{ (I has a winning strategy in } (F_n)_s).$$

Here $D_s = \{x \in A^{\mathbb{N}} \mid s \hat{\ } x \in D\}$ for a subset D of $A^{\mathbb{N}}$.

And if W_η has been defined for all $\eta < \xi$, we let

$$x \in C_n^\xi \Leftrightarrow \forall \text{ even } k \ (x|_k \in \bigcup_{\eta < \xi} W_\eta \cup T_n),$$

and set

$$s \in W_\xi \Leftrightarrow |s| \text{ is even} \ \& \ \exists n \text{ (I has a winning strategy to play in } (C_n^\xi)_s).$$

Notice that each C_n^ξ is closed and show that (1) $s \in \bigcup_\xi W_\xi \Rightarrow$ I has a winning strategy in B_s and (2) $\emptyset \notin \bigcup_\xi W_\xi \Rightarrow$ II has a winning strategy in B .

3. EXERCISE

Show that all Σ_1^1 games are determined if and only if all Π_1^1 games are determined.

4. EXERCISE

Let X be a Polish space and \leq a Π_1^1 partial preorder (i.e., a quasiorder) on X . Suppose $A \subseteq X$ is a Σ_1^1 set such that $\leq|_A$ is total, i.e., $\forall x, y \in A \ (x \leq y \text{ or } y \leq x)$. Show that there is a Borel set $B \supseteq A$ such that $\leq|_B$ is total.